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Current reversals in ratchets driven by trichotomous noise

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The colored three-level Markovian noise-driven nonequilibrium dynamics of overdamped Brownian particles in a spatially periodic asymmetric potential (ratchet) is investigated. An explicit second-order linear ordinary differential equation for the stationary probability density distribution is obtained for the process. In the case of a piecewise linear potential with an additive three-level (trichotomous) noise the exact formula for the stationary current is presented. The dependence of the current reversals on the noise parameters is investigated in detail and illustrated by a phase diagram. Asymptotic formulas for the current for various limits of the noise parameters are found and compared with the results of other authors. Applications to the fluctuation-induced separation of particles are also discussed.

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I. INTRODUCTION

Within the past few years there has been considerable interest in the problem of noise-induced transport in spatially periodic structures called ratchets (for reference surveys see [1,2]). The initial motivation in this field has come from the cell biology, in particular the study of the mechanism of vesicles transport inside eukariotic cells, via motor proteins along microtubules [2,3]. It has been argued in [3] that a ratchet (a Brownian motor) could extract energy from nonequilibrium fluctuations even if their mean value is equal to zero. Later on new systems with the same underlying ideas for transportation were proposed, (e.g. chemically driven motility of enzymatic Brownian particles [4], phase separation engines [5], growth of surfaces [6], and rectification in superconducting rings [7]). There are several categories of models for stochastic ratchets [1–3,8–12]. It should be noted that the dynamics in ratchet structures with its inherent spatial asymmetry generally exhibits a rich complexity, such as the occurrence of multiple current reversals and multipeaked current characteristics [1,2]. A particularly appealing feature of Brownian motors is their ability to separate particles of different friction strength or mass [1]. It is well known that the net current in a periodic ratchet potential fluctuating randomly between a flat and a nonflat state is always biased in one direction, independent of the correlation time of the fluctuation [10,13]. In some cases current reversal (CR) could be observed, i.e., the current changed its direction in certain parameter regions of the model [9,11,14–27]. Millonas and Dykman have discussed the generation of CR in a stationary periodic potential induced by a Gaussian force noise with a nonwhite power spectrum [14]. Chauwin, Ajdari, and Prost [9] have suggested that CR can be obtained in the two-state ratchet model if the long arm of the ratchet is kinked. Bier and Astumian have also found CR in a fluctuating three-state model [15]. Doering, Horsthemke, and Riordan [11] introduced a kangaroo process as the driving force and found that CR depends on the flatness (the ratio of the fourth moment to the square of the second moment) of the noise. Later, Mielke [22] developed a method that allows us to calculate the current for a large class of processes including those discussed in [11], again in the case of a sawtooth potential (piecewise linear potential), and found several other cases where CR occurs. Similarly, in [16] it has been shown that a periodic force can cause a CR depending on its amplitude and frequency.

Bartussek, Reimann, and Hänggi [20] have presented CR in a correlation ratchet driven by both an additive Gaussian white and an additive Ornstein–Uhlenbeck noise. Depending on the choice of the ratchet potential, CR may occur at a specific value of the correlation time. For an inertia ratchet a CR can be evoked by modifying the mass of the particles [21,24].

The effect of CR in combination with the stationary carrier density has been considered in [25,26], where the diffusing particles were interacting and the ensuing ratchet current described a collective dynamics.

In [17–19] calculations are presented for a three-level Markovian stochastic force and approximations for the mean current have been carried out for the limits of slow and fast noise. It has been shown that the direction of the current may depend on the correlation time of the noise as well as on the flatness parameter. These models are potentially very useful, because CR could lead to a more efficient fluctuation-induced separation of particles [9,17]. Nevertheless, most of the results have been obtained by numerical methods or for limits of slow and fast noises. There are almost no exact results for correlation ratchets, enabling us to quantitatively evaluate the values of the noise parameters corresponding to CRs for concrete models, or giving sufficient and necessary conditions for their existence. This is caused, first and foremost, by the fact that even simple model ratchets display a rich variety of behaviors that vary remarkably with the system parameters. Capturing the full range of these possibilities—and the transitions between them—as several

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parameters change, is quite difficult with numerical solutions alone.

In this paper we consider one-dimensional overdamped dynamical systems determined by a first-order differential equation with a periodic potential and an additive noise term composed of a trichotomous process, which is a three-level stationary telegraph process characterized by three parameters: amplitude $a_0 \in (0,\infty)$, correlation time $\tau_c \in (0,\infty)$, and flatness $\varphi \in (1,\infty)$ [28,29]. In order to get the results in a closed form for all values of the noise parameters, the noise is applied to piecewise linear (sawtooth) potentials, which have been considered as applicable to scientific and engineering problems as good approximations to potentials met in the real world [30].

The purpose of this paper is to provide exact analytical results for the stationary current J over extended noise parameter regimes for the system. Interpreting the qualitatively different shapes of the dependence of J on the correlation time τ_c as different phases in the phase space of the parameters φ and a_0, we have constructed comprehensive phase diagrams to demonstrate the noise-induced transitions. Here we succeeded in reaching the exact conditions which bring forth CR.

The structure of the paper is as follows. In Sec. II the model and exact differential equation for the stationary probability density are presented. The current for periodic potentials in the addiabatic limit is investigated. In Sec. III a dynamical system with a periodic sawtooth potential is considered. The exact stationary current is found. In Sec. IV the behavior of the current at different limits, such as the slow noise limit, large amplitude limit etc., is analyzed. In Sec. V the current reversals are subjected to a closer consideration. The dependence of CR on the noise parameters is investigated and comprehensive phase diagrams are presented. Section VI contains some concluding remarks.

II. TRICHOTOMOUS MARKOVIAN NOISE

Here we explicate the idea of dichotomous noise further to a symmetric three-level random telegraph process $f(t)$ called the *trichotomous process* [28]. This is a random stationary Markovian process that consists of jumps between three values $a=a_0,0,-a_0$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities

$$P_s(a_0)=P_s(-a_0)=q, \quad P_s(0)=1-2q. \tag{1}$$

The transition probabilities between the states $f(t)=\pm a_0$ and 0 can be obtained as follows:

$$P(\pm a_0,t+\tau|0,t)=P(-a_0,t+\tau|a_0,t)=P(a_0,t+\tau|-a_0,t)$$
$$=q(1-e^{-\nu\tau}),$$
$$P(0,t+\tau|\pm a_0,t)=(1-2q)(1-e^{-\nu\tau}), \tag{2}$$
$$\tau>0,\ \ 0<q<1/2,\ \ \nu>0.$$

The process is completely determined by Eqs. (1) and (2). The mean value of $f(t)$ and the correlation function are

$$\langle f(t)\rangle=0, \tag{3}$$

$$\langle f(t),f(t')\rangle=\langle a^2\rangle e^{-\nu|t-t'|}=2qa_0^2e^{-\nu|t-t'|}.$$

It can be seen that the switching rate ν is the reciprocal of the noise correlation time

$$\nu=1/\tau_c.$$

The noise intensity is

$$\sigma^2=2\int_0^\infty\langle f(t+\tau),f(t)\rangle d\tau=4qa_0^2/\nu. \tag{4}$$

The flatness parameter φ proves to be a very simple expression of the probability q

$$\varphi=\langle f^4(t)\rangle/\langle f^2(t)\rangle^2=1/(2q). \tag{5}$$

Next, we consider an overdamped motion in an asymmetric periodic potential $U(x)$ with the period L. The process is driven by the trichotomous noise $f(t)$. The motion is described by the stochastic differential equation

$$\kappa\frac{dx}{dt}=h(x)+f(t),\ \ h(x)\equiv-\frac{dU(x)}{dx}, \tag{6}$$

where κ is the viscous friction strength. By applying a scaling of the form

$$\tilde{x}=x/L,\ \ \tilde{t}=t/t_0,\ \ \tilde{f}=fL/U_0,\ \ V(\tilde{x})=U(x)/U_0$$

we get a dimensionless formulation of the dynamics with the potential V with the property $V(\tilde{x})=V(\tilde{x}+1)$. By the choice $t_0=\kappa L^2/U_0$ the dimensionless friction coefficient turns to unity. The rescaled noise parameters are given by

$$\tilde{\nu}=\kappa L^2\nu/U_0,\ \ \tilde{a}_0=La_0/U_0. \tag{7}$$

From now on we shall use only the dimensionless dynamics and omit the tildes. The dynamics reads

$$\frac{dx}{dt}=h(x)+f(t),\ \ h(x)\equiv-\frac{dV(x)}{dx}. \tag{8}$$

The corresponding composite Fokker–Planck master equation for our problem is

$$\frac{\partial}{\partial t}P_n(x,t)=-\frac{\partial}{\partial x}\{[h(x)+a_n]P_n(x,t)\}+\sum_m U_{nm}P_m(x,t), \tag{9}$$

with $P_n(x,t)$ denoting the probability density for the combined process (x,a_n,t); $n,m=1,2,3$; $a_1\equiv-a_0$, $a_2\equiv0$, $a_3\equiv a_0$ and

$$\mathbf{U}=\nu\begin{pmatrix} q-1 & q & q\\ 1-2q & -2q & 1-2q\\ q & q & q-1\end{pmatrix}. \tag{10}$$

The stationary current J is then evaluated via the current densities

$$j_n(x)=[h(x)+a_n]P_n^s(x),$$

$$J=\sum_n j_n(x), \tag{11}$$

where $P_n^s(x)$ is the stationary probability density for the state (x,a_n). It follows from Eq. (9) that the current J is constant. We shall assume that

$$\max_x h(x) > |\min_x h(x)|.$$

The following characteristic regions for a_0 can be discerned:

(i) There is no current if $0 < a_0 < |\min_x h(x)|$, as there is a stationary stable point for any state n.

(ii) In case of $|\min_x h(x)| < a_0 < \max_x h(x)$ there exists at least one stationary stable point for $f(t) = -a_0$, the motion to the left is switched off and the current J is positive.

(iii) In case of $a_0 > \max_x h(x)$ the stochastic process $f(t)$ can, although it should not, induce a reversal of the current. Now we shall discuss this case in some detail.

For the calculation of the stationary probability density in the x space $P(x) = \Sigma_n P_n^s(x)$ and the stationary current $J =$ const the results of [31] can be applied. Notably, it is shown there that if a process $x(t)$ satisfies the stochastic differential equation (8), where $f(t)$ is a generalized random telegraph process, the stationary probability density $P(x)$ is a solution of the operator equation

$$J - h(x)P(x) = \nu \langle a \hat{L}_a^{-1} \rangle P(x). \tag{12}$$

The angular brackets $\langle\rangle$ mean averaging over the values of the random variable a and the operator $\hat{L}_a^{-1}$ is the inverse of the operator $\hat{L}_a$ defined by

$$\hat{L}_a \psi(x) := \nu \psi(x) + \frac{d}{dx}[(h(x)+a)\psi(x)]. \tag{13}$$

In our Eq. (8) the random variable a takes the values a_0, $-a_0$ with the probability q and the value 0 with the probability $1-2q$. For the stationary probability density $P(x)$ corresponding to Eq. (8) the following second-order differential equation can be obtained from Eq. (12):

$$\begin{aligned} J(h'(x)+\nu) - \nu h(x)P(x) + \frac{d}{dx}[(a_0^2 - h^2(x))P(x)] \\ + \frac{d}{dx}\left\{ \frac{h(x)}{h'(x)+\nu}\left[(h'(x)+\nu)J - \nu h(x)P(x) \right.\right. \\ \left.\left. + \frac{d}{dx}[(a_0^2 - h^2(x))P(x)] \right]\right\} \\ = (1-2q)\nu a_0^2 \frac{d}{dx}\left[\frac{P(x)}{\nu + h'(x)} \right], \end{aligned} \tag{14}$$

where

$$h'(x) \equiv \frac{d}{dx} h(x).$$

In the case of $q = 1/2$ (a dichotomous noise) the last term vanishes and Eq. (14) is satisfied by every solution of the equation

$$J(h'(x)+\nu) - \nu h(x)P(x) + \frac{d}{dx}[(a_0^2 - h^2(x))P(x)] = 0.$$

The latter corresponds to Eq. (8) in the case where $f(t)$ is a dichotomous noise. This has been investigated in detail by several authors [11,32]. In order to proceed further with the calculations in the case of a trichotomous noise it is suitable to impose periodic boundary conditions on the stationary probability density

$$P(x) = P(x+1),$$

which converts Eq. (12) to the form

$$J - h(x)P(x) = q a_0 \int_x^{x+1} dy\, P(y) \frac{d}{dx}\left[\frac{\chi(y)}{(\chi(1)-1)\chi(x)} - \frac{\phi(y)}{(\phi(1)-1)\phi(x)} \right], \tag{15}$$

where

$$\phi(x) := \exp\left(\nu \int_0^x \frac{dy}{h(y)+a_0} \right), \quad \chi(x) := \exp\left(\nu \int_0^x \frac{dy}{h(y)-a_0} \right). \tag{16}$$

The constant stationary current J can be specified by the application of the normalization condition to $P(x)$

$$\int_0^1 P(x)dx = 1. \tag{17}$$

Thus, a combination of Eqs. (15)–(17) with Eq. (8) yields the following relation between the average of the particle velocity $\langle dx/dt \rangle$ and the current J:

$$\langle dx/dt \rangle = \langle h(x) \rangle = \int_0^1 h(x)P(x)dx = J. \tag{18}$$

It is remarkable that in the case of a trichotomous noise the stationary probability density $P(x)$ corresponding to Eq. (8) is determined by a relatively simple second-order linear ordinary differential equation and the behavior of $P(x)$ can be investigated by the general theory of such equations. Unfortunately exact solutions of Eq. (14) can be obtained but in a few cases. The simplest example of such is the so-called adiabatic limit $\nu \to 0$. For simplicity, we assume that $V(x)$ has only one minimum at $x = d$ $(0 \leq x < 1)$. If $a_0 > \max_x h(x)$, then the stationary probability density is given by

$$P(x) = \frac{C_+}{a_0 + h(x)} + \frac{C_-}{a_0 - h(x)} + (1-2q)\sum_k \delta(x-d-k), \tag{19}$$

where the constants $C_\pm$ are determined by

$$C_\pm \int_0^1 \frac{dx}{a_0 \pm h(x)} = q \tag{20}$$

and $d+k$ in the arguments of the δ function denote the locations of the minima of $V(x)$. This leads to the following expression for the current:

$$J = \int_0^1 \frac{h(x)[a_0(C_+ + C_-) + h(x)(C_- - C_+)]}{a_0^2 - h^2(x)} dx. \tag{21}$$

It is easy to ascertain that in the limit $a_0 \to \max_x h(x)$ the current is $J = C_+$ and therefore is positive. If $a_0 \gg \max_x h(x)$ then J tends to zero and the following asymptotic equation holds true:

$$J = \frac{2q}{a_0^2} \int_0^1 h^3(x)\,dx + O\left(\frac{1}{a_0^4}\right).$$

The effect of the steepness of the slopes of the potential V is stressed by the integral of h^3, vanishing if the potential is symmetric. Trichotomous noise is a particular case of the kangaroo process, at which the leading-order correction of the current has been investigated by Doering, Horsthemke, and Riordan [11] in the white noise limit. It is notable that there the leading-order correction of J is also proportional to an integral of h^3.

In the case of $|\min_x h(x)| < a_0 < \max_x h(x)$ it follows from Eqs. (9)–(11) that

$$\lim_{\nu \to 0} J = C_+ > 0. \tag{22}$$

Equation (20) shows that as a_0 grows, the current grows monotonically.

III. EXACT SOLUTION FOR A SAWTOOTH POTENTIAL

The integral equation (15) can be solved exactly for some special forms of the potential $V(x)$ only. We present an analysis of the system of Eq. (8) for a piecewise linear sawtooth-like potential

$$V(x) = \begin{cases} -(x-d)/d, & x \in (0,d) \mod 1, \\ (x-d)/(1-d), & x \in (d,1) \mod 1, \end{cases} \tag{23}$$

where $d \in (0,1)$ determines the asymmetry of the potential, which is symmetric if $d = 1/2$. The space being left–right symmetric, we may confine ourselves to the case $d \leqslant 1/2$. As our starting equation (15) has been derived at the assumption that $V(x)$ is differentiable at every point, we have to consider the sawtooth potential as a limit case of a smooth potential, so that

$$h(d+k) = h(k) = 0, \tag{24}$$

with k being an integer. Such a potential can characterize, for example, the force

$$\frac{dV(x)}{dx} = \frac{1}{2d(1-d)} \times \left[\tanh\left(\frac{x - d - \epsilon \ln\sqrt{d/(1-d)}}{\epsilon}\right) - 1 + 2d\right],$$

where $0 < x < 1$ and $\epsilon > 0$. For a small $\epsilon/d \ll 1$ the shape of the corresponding potential is close to that of the sawtooth, with $h(d) = 0$.

The force $h(x)$ being periodic, the stationary distribution $P(x)$ as a solution of Eq. (15) is also periodic and it suffices to consider the problem in the interval $[0,1)$. The force corresponding to the potential, Eq. (23), is

$$h(x) = -\frac{dV(x)}{dx} = \begin{cases} b := 1/d, & x \in (0,d), \\ -c := -1/(1-d), & x \in (d,1), \\ 0, & x = 0,\ x = d. \end{cases} \tag{25}$$

Evidently, from Eq. (14) the following solution can be obtained:

$$P(x) = \tilde{P}(x) + \rho\,\delta(x-d) \tag{26}$$

with $\rho = \text{const}$ and

$$\tilde{P}(x) = \begin{cases} C_1 e^{\lambda_{11} x} + C_2 e^{\lambda_{12} x} + J/b, & x \in (0,d), \\ G_1 e^{\lambda_{21} x} + G_2 e^{\lambda_{22} x} - J/c, & x \in (d,1), \end{cases}$$

where

$$\lambda_{1i} = -\frac{\nu}{b(a_0^2 - b^2)}(q a_0^2 - b^2 \pm a_0 \eta),$$

$$\lambda_{2i} = \frac{\nu}{c(a_0^2 - c^2)}(q a_0^2 - c^2 \pm a_0 \gamma),$$

$$\eta := \sqrt{(1-2q)b^2 + q^2 a_0^2}, \quad \gamma := \sqrt{(1-2q)c^2 + q^2 a_0^2}$$

and $i = 1,2$ with $i = 1$ corresponding to the sign $+$ and $i = 2$ to the sign $-$, respectively. The current J and the five constants ρ, C_i, G_i $(i = 1,2)$ are determined by Eqs. (15) and (17). By substituting Eq. (26) in them we get a nonhomogeneous system of six linear algebraic equations. Hence, the problem is solved at that and the evaluation of the current can be handled by linear algebra. The exact form of the stationary current is

$$J = \frac{4\nu q \gamma \eta b c [A(b,c) - A(c,b)]}{A(c,b)B(c,b) + A(b,c)B(c,b) - 4\nu q \gamma \eta (b-c)[A(b,c) - A(c,b)]}, \tag{27}$$

where

$$A(b,c) := a_0 \gamma \eta \{ bc(\alpha_1 + \alpha_2) - (1-2q)[b(\beta_1 + \beta_2) + c(\alpha_1 + \alpha_2)] \} + \gamma c(\alpha_2 - \alpha_1)[(1-2q)(b^2 - q a_0^2) + q a_0^2 b] - \eta b(1-2q)(\beta_2 - \beta_1)(q a_0^2 - c^2),$$

$$B(b,c) := \gamma \eta \{ 2 q a_0^2 b(\beta_1 + \beta_2) + (1-2q)[b(\beta_1 + \beta_2)(c - 2 q a_0^2/c) - c(\alpha_1 + \alpha_2)(b - 2 q a_0^2/b)] \} + \eta a_0 (\beta_2 - \beta_1)\{ b[c^2(1-2q) + 2 q^2 a_0^2] - (1-2q) b[(1-3q)c + 2 q^2 a_0^2/c] \} + \gamma a_0 c(1-2q)(\alpha_2 - \alpha_1)[(1-3q)b + 2 q^2 a_0^2/b], \tag{28}$$

and

$$\alpha_i := \exp(\lambda_{1i}/b) - 1, \quad \beta_i := \exp(-\lambda_{2i}/c) - 1, \quad i = 1,2.$$

It should be noted that in the case of $a_0 > b$ and the finite parameters the denominator in Eq. (27) is always positive. Obviously, if the potential is symmetric $(b=c)$, there is no current in the stationary state.

It is also of interest to consider the parameter ρ, characterizing the probability that the state of the system coincides with the deterministic stationary state $x=d$ (stationary stable point at the absence of noise). This reads

$$\rho = \frac{(1-2q)}{a_0} \frac{a_0^2[b\gamma(\alpha_2-\alpha_1)+c\eta(\beta_2-\beta_1)][bA(c,b)+cA(b,c)]-[A(b,c)-A(c,b)]^2}{A(c,b)B(c,b)+A(b,c)B(b,c)-4\nu q\gamma\eta(b-c)[A(b,c)-A(c,b)]}. \tag{29}$$

It follows from Eq. (29) that in the case of a dichotomous noise $(q=1/2)$ the parameter ρ vanishes for any a_0, ν, and d. It is evident from physical considerations that $0 \leqslant \rho \leqslant 1 - 2q$ should always be valid.

Substituting Eqs. (25) and (26) into Eqs. (17) and (18) we can see that the range of J is bounded

$$(1-\rho)b > J > -c(1-\rho).$$

Hence, the current is greater than $-c$ and less than b for all values of ν, a_0, and φ.

When investigating by Eq. (27) the dependence of J on the correlation time τ_c, four different types of the graphs $J(\nu)$ emerge. In Fig. 1 these four types are represented as depending on the parameters q and a_0. (i) If $q=0.3$ and $a_0=5.2$, no reversals of the current are met and with increasing ν the current decreases sigmoidally and monotonically to 0. (ii) If $q=0.3$ and $a_0=7$, we obtain a nonmonotonic behavior, where the ratchet current reaches the minimum and maximum at a finite ν. The current is not reversed either. (iii) If $q=0.3$ and $a_0=9$, the current exhibits two reversals of the direction at increasing values of ν. (iv) At $q=0.1$ and $a_0=9$ we have a single reversal from positive to negative and finally J approaches zero from the negative side. We interpret these four qualitatively different shapes of $J(\nu)$ as different phases in the phase space (q,a_0).

It should be noted that for other model systems the same four phases [i.e., the typical forms of the graph of $J(\nu)$] have been reached by several authors by means of numerical calculation [17–19].

The behavior of J at different asymptotics and the conditions of the occurrence of the phases will be considered in Secs. IV and V.

IV. ASYMPTOTIC REGIMES

Here the asymptotic regimes following from Eq. (27) will be studied.

A. The long-correlation-time limit

At the adiabatic limit $\nu \to 0$ Eq. (27) takes the form

$$J \approx \frac{2q(b^2-c^2)}{a_0^2-(b-c)^2} > 0, \tag{30}$$

that tends monotonically to zero as $a_0 \to \infty$ or $q \to 0$. Equation (30) follows also immediately from Eqs. (20) and (21). For ρ we can get

$$\rho \approx 1 - 2q. \tag{31}$$

This result can be understood intuitively by means of Eq. (1): the random variable a takes value 0 for a sufficiently long time to allow the deterministic stationary state be formed.

B. The white noise limit

In the trichotomous δ-correlated limit (i.e., $\nu \to \infty$, $a_0 \to \infty$, so that $\sigma^2 = 4qa_0^2/\nu$ is finite) Eqs. (27) and (29) reduce, respectively, to

$$J \approx \frac{8(b^2-c^2)e^{2/\sigma^2}}{\nu\sigma^6(e^{2/\sigma^2}-1)^2}(2-\varphi), \quad \varphi = \frac{1}{2q}, \tag{32}$$

$$\rho \approx (1-2q)\frac{(b+c)e^{2/\sigma^2}}{\nu q\sigma^2(e^{2/\sigma^2}-1)}. \tag{33}$$

The current and the parameter ρ in this limit are proportional to the noise correlation time that in these cases is a measure of the distance from equilibrium. The current in Eq. (32) has a factor dependent on the noise statistics via the flatness parameter φ. If the statistics of the trichotomous noise corresponds to $\varphi > 2$, i.e., $q < 1/4$, the sign of the current changes in complete accordance with the results of [11] where the general kangaroo process is considered. It should also be noted that ρ decreases monotonically to the value $(1-2q)bc/(2\nu q)$ as the noise intensity grows. Current J takes an extremum at $\sigma^2 \approx 0.7765$.

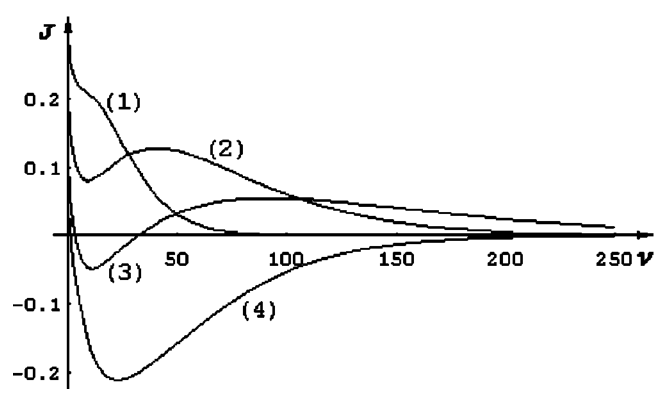

FIG. 1. The current J vs the switching rate ν. The curves (1)–(4) correspond to the following parameters: (1): $q=0.3$, $a_0=5.2$; (2): $q=0.3$, $a_0=7$; (3): $q=0.3$, $a_0=9$; (4): $q=0.1$, $a_0=9$. In the cases (3) and (4) current reversals occur.

C. The large-flatness limit

In the case of $q\to 0$ ($\varphi\to\infty$) the current and ρ are found to be

$$J\approx \nu q\{[e^{\nu/c(a_0-c)}-e^{-\nu/b(a_0+b)}]^{-1} -[e^{\nu/b(a_0-b)}-e^{-\nu/c(a_0+c)}]^{-1}\},\quad \rho\approx 1. \tag{34}$$

We can see that the current is reciprocal to the flatness parameter. If $a_0\leqslant a_{0cr}\equiv b+c$, then J is positive at any ν. If the noise amplitude exceeds a_{0cr} then the current reverses to negative at $\nu>\nu^*$. The point of reversal ν^*, being a solution of a transcendental equation $J(\nu^*)=0$, can in a general case be found by a numerical calculation. Some of its properties can be analyzed analytically, though. As the noise amplitude grows within the region $a_0>b+c$, the parameter ν^* decreases monotonically from infinity to zero. If $a_0\gg b+c$, the following asymptotic formula is valid:

$$\nu^*\approx 2(b+c)^2/a_0^2. \tag{35}$$

In the vicinity of the critical point $q(b+c)\ll a_0-(b+c)\ll 1$, the following asymptotic formula can be used:

$$\nu^*\approx \frac{b(a_0^2-b^2)}{2a_0}\ln\left(\frac{1}{a_0-bc}\right). \tag{36}$$

D. The large-amplitude limit

For $a_0\to\infty$ and for fixed ν and q, i.e., for the case of a very large noise intensity, the current saturates at the value

$$J\approx -\frac{1-2q}{2q\nu}[b^2(1-e^{-2\nu q/b^2})-c^2(1-e^{-2\nu q/c^2})]. \tag{37}$$

It can be easily seen that $J<0$ at any values of the parameters $q\in(0,1/2)$ and $\nu\in(0,\infty)$. This result is not inconsistent with Eq. (30)—it is just that the current reversal occurs at the switching rate $\nu=0$. Obviously, J tends to zero as $\nu q\to\infty$ or as $\nu q\to 0$. Consequently, there occurs a minimum of $J(\nu)$. For $b\gg c$, the minimum of $J(\nu)$ occurs at $\nu_m\approx cb/\sqrt{2}q$ and $J_{min}\approx -(1-2q)$. In a general case ν_m can be found by the following transcendental equation ($x=2q\nu_m$):

$$(x+b^2)e^{-x/b^2}=(b^2-c^2)+(x+c^2)e^{-x/c^2}. \tag{38}$$

Parameter ρ behaves asymptotically as

$$\rho\approx \frac{1-2q}{2\nu q}[b(1-e^{-2\nu q/b^2})+c(1-e^{-2\nu q/c^2})]. \tag{39}$$

We can see that, as the correlation time grows, $\nu q\to 0$, the share of the particles concentrated in the minimum of the potential grows monotonically from zero to $1-2q$.

It is quite remarkable that in case of fixed φ and τ_c the current saturates to a finite value at great noise amplitudes. This counterintuitive result is due to both an effective inhomogeneous diffusion, which becomes more homogeneous with increasing a_0, and a so-called ''flashing barrier'' effect as stated in [17]. Let us look into the latter statement more closely with the assumptions that $a_0^2\gg b^2/q^2(1-2q)$ and $\nu\ll 2q(1-2q)a_0^2$. For these assumptions the probability distributions $P_{1,3}^s(x)$ at the noise source states $\pm a_0$ are, evidently, homogeneous and within the interval (0,1) the center of mass is at $y_0=1/2$. Let the noise in the initial time be at the state $a=0$. The first time when the noise turns to either $a=a_0$ or $a=-a_0$ is denoted by t_0. The center of mass at time t_0 is located at y. It is easy to find that the center of mass is shifted by $\Delta y\equiv y-y_0$

$$\Delta y=\begin{cases}-(b-c)/2bc, & t_0\geqslant 1/c^2,\\ 1/2b^2-t_0(1-c^2t_0/2), & 1/b^2\leqslant t_0\leqslant 1/c^2,\\ -t_0^2(b^2-c^2)/2, & t_0\leqslant 1/b^2.\end{cases}$$

In the case of a trichotomous noise the probability $W(t_0)$ that in a certain time interval $(0,t_0)$ the transitions $a=0\to a=\pm a_0$ do not occur, is given by $W(t_0)=\exp(-2q\nu t_0)$. The probability that such a transition occurs within the time interval (t_0,t_0+dt_0) is $2q\nu dt_0$. Consequently,

$$\langle\Delta y\rangle=2q\nu\int_0^\infty e^{-2q\nu t_0}\Delta y(t_0)dt_0. \tag{40}$$

Considering that the average number of transitions per unit of time into the 0 state is $2q\nu(1-2q)$, we obtain

$$J=2q\nu(1-2q)\langle\Delta y\rangle.$$

When calculating by Eq. (40) the mean value of the shift of the center of mass we can reach the earlier result Eq. (37). Thus, at sufficiently large noise amplitudes and the correlation times satisfying $\nu\ll 2q(1-2q)a_0^2$ the behavior of the current is determined only by the ''flashing barrier'' effect.

E. The fast noise limit

In the fast noise limit we allow ν to become large, holding all other parameters fixed, and use ν^{-1} as the smallness parameter in our expansion. Thus, in the large ν limit the current is exponentially small:

$$J\sim\begin{cases}\nu e^{\lambda_{22}/c}, & \alpha_2>\beta_2,\\ -\nu e^{-\lambda_{12}/b}, & \beta_2>\alpha_2,\end{cases} \tag{41}$$

and J tends to zero as $\nu\to\infty$. It is remarkable that for small correlation time the current cannot be expanded into a power series with respect to $1/\nu$. The current is positive for $\alpha_2>\beta_2$ and negative for $\beta_2>\alpha_2$. The latter can happen only if the flatness is greater than 2, i.e., if $q<1/4$, and a_0 is greater than a critical value a_c

$$a_c^2=\frac{(1-2q)(b^2+c^2)}{1-4q}\left[1+\sqrt{1-\frac{(b^2-c^2)^2(1-4q)}{(1-2q)^2(b^2+c^2)^2}}\right]. \tag{42}$$

Obviously, $a_c\geqslant b+c$, where the sign of equality corresponds to $q=0$.

In general, at large values of ν the parameter ρ stabilizes at a finite value. The expression for this being cumbersome, we bring it but for the limit of $\nu\gg qa_0\to\infty$

$$\rho\approx(1-2q)\frac{bc}{4q^2a_0^2}. \tag{43}$$

Thus, at a small correlation time and large noise amplitudes $\rho\to 0$.

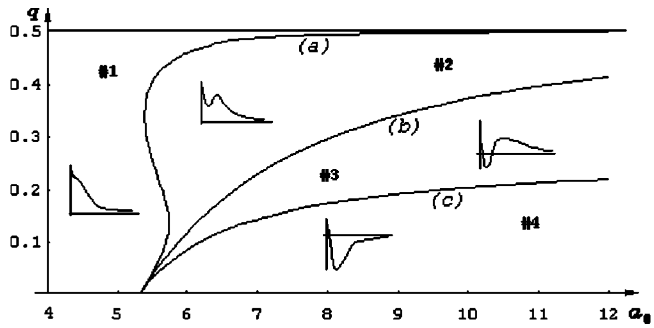

FIG. 2. The (q,a_0) phase diagram for the dependence of the stationary current J on ν in the case of $d=0.25$. The shape of the function $J(\nu)$ for the different domains formed by the curves (a)–(c) are sketched. Current reversals occur in domain Nos. 3 and 4. The curves (a), (b), and (c) are determined by Eqs. (45), (46), and (42), respectively.

F. The dichotomous Markovian noise

For dichotomous noises $q=1/2$, the exact stationary current and the parameter ρ are

$$J=\frac{\nu bc(\alpha_2-\beta_2)}{a_0^2 bc\alpha_2\beta_2-\nu(b-c)(\alpha_2-\beta_2)}>0, \quad \rho=0, \quad (44)$$

where

$$\alpha_2=\exp[\nu/(a_0^2-b^2)]-1,$$

$$\beta_2=\exp[\nu/(a_0^2-c^2)]-1.$$

As the correlation time decreases, J decreases monotonically from $J=(b^2-c^2)/[a_0^2-(b-c)^2]$ to zero. No current reversal occurs. Equation (44) accords with the expression in [11] for the stationary current in the case of a dichotomous noise, if its general smooth potential $V(x)$ is replaced by our potential Eq. (23) and the second derivative of $V(x)$ is substituted to the delta function: $d^2V(x)/dx^2\to(b+c)\delta(x-d)$, respectively.

V. REVERSALS OF NOISE-INDUCED CURRENT

Next, we shall consider the most general properties of the stationary current $J(\nu)$ in the phase space of the noise parameters q and a_0. Proceeding from Eqs. (27) and (28) we can distinguish between four domains in the two-dimensional phase space (q,a_0) (see Fig. 2).

(1) $b<a_0<a_1(q)$. In this domain the current $J(\nu)$ is positive and decreases monotonically to zero as ν increases. The ratchet model with dichotomous noise belongs here as a limit case. The boundary of the domain (a) is given by the system of transcendental equations

$$\left(\frac{\partial}{\partial\nu}J(\nu)\right)\bigg|_{a_0=a_1(q)}=0,$$

$$\left(\frac{\partial^2}{\partial\nu^2}J(\nu)\right)\bigg|_{a_0=a_1(q)}=0. \quad (45)$$

(2) $a_1(q)<a_0<a_2(q)$. The stationary current $J(\nu)>0$ is bimodal and reaches a local minimum and a local maximum at a finite ν. The curve (b) where $a_0=a_2(q)$, is given by the system of transcendental equations

$$A(b,c)|_{a_0=a_2(q)}=A(c,b)|_{a_0=a_2(q)},$$

$$\left(\frac{\partial}{\partial\nu}A(b,c)\right)\bigg|_{a_0=a_2(q)}=\left(\frac{\partial}{\partial\nu}A(c,b)\right)\bigg|_{a_0=a_2(q)}, \quad (46)$$

where $A(b,c)$ has been defined in Eq. (28).

(3) $a_2(q)<a_0<a_c(q)$. The current exhibits a double reversal of the direction for increasing values of ν. The current starts from a positive value, decreasing to a negative local minimum, next it grows, attaining a positive maximum, and then J approaches zero as $\nu\to\infty$. The curve (c) where $a_0=a_c(q)$, is given by the explicit result of Eq. (42).

(4) In this domain $a_0>a_c(q)$ the flatness of the noise is greater than 2. A single current reversal occurs, and $J\to-0$ as $\nu\to\infty$.

It should be noted that though the phase boundary (c) can be described by an exact analytical formula, the boundary lines (a) and (b) cannot. The latter can be expressed from Eqs. (45) and (46) by numerical methods or by using approximate equations. For (b), the following approximate equation seems acceptable:

$$a_2^2(q)\approx\frac{2qb^2}{1-2q}\left[1+\frac{3}{2}\left(\exp\frac{2\sqrt{2}c}{3b}-1\right)\right]+(b+c)^2. \quad (47)$$

According to numerical calculations with various values of the system parameters $(0.01\leqslant d\leqslant 0.49,\ 0.001\leqslant q\leqslant 0.4995)$ the application of Eq. (47) does not cause the relative error to exceed 1%.

At very large values of the flatness parameter, when $q\to 0$, all three phase boundaries approach each other

$$a_1(q)\approx a_2(q)\approx a_c(q)=(1+q)(b+c)+O(q^2). \quad (48)$$

Notably, unlike $a_2(q)$ and $a_c(q)$, the function $a_1(q)$ is not always monotonically growing but may have a local maximum and a minimum. It is interesting to note that in the domains (1–3) ρ is a monotonically decreasing function that stabilizes at a nonzero value as ν grows, but in the domain (4) ρ is nonmonotonic and has a minimum at a certain finite value of ν.

For the calculation of the current reversal points ν^*, $J(\nu^*)=0$, by numerical calculation we propose the transcendental equation $A(b,c)=A(c,b)$, i.e.,

$$\gamma[\eta(\alpha_1+\alpha_2)+qa_0(\alpha_2-\alpha_1)]$$
$$=\eta[\gamma(\beta_1+\beta_2)+qa_0(\beta_2-\beta_1)]. \quad (49)$$

The dependence of these points on the noise amplitude a_0 and the flatness parameter $\varphi=1/2q$ is illustrated in Fig. 3. As the noise amplitude a_0 grows, a current reversal first appears at the noise amplitude value $a_0=a_2(q)$. When the flatness parameter is less than 2, the growth of a_0 always causes double reversal. The current changes its sign at the two noise correlation time values $\tau_1^*=1/\nu_1^*$ and $\tau_2^*=1/\nu_2^*$. An increase of the amplitude $(a_0\to\infty)$ causes the first solution of Eq. (49) ν_1^* to drop monotonically to zero, while the second

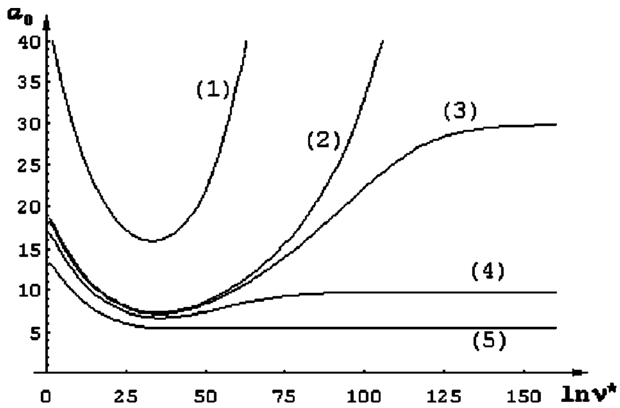

FIG. 3. The noise amplitude a_0 vs log of the switching rate ν^* corresponding to the current reversal points, $J(\nu^*)=0$, for several flatness values. The curves (1)–(5) correspond to the values of $d=0.25$ and $q=0.45$, 0.255, 0.245, 0.2, and 0.01, respectively. The minima of the curves lie at the noise amplitude value $a_0=a_2(q)$ [see Eqs. (46)]. As $q\to 1/2$, the value of ν^* corresponding to $a_2(q)$ saturates at ν_0^* [Eq. (38) with $x=\nu_0^*$]. If $q<1/4$, then at a fixed noise amplitude $a_0>a_c$ [Eq. (42)] only one current reversal point ν^* occurs.

solution ν_2^* increases monotonically to infinity. As the flatness parameter decreases, the critical amplitude $a_2(q)$ increases; ν_1^*, corresponding to a fixed amplitude, increases monotonically, but ν_2^* monotonically decreases at a decreasing φ. In the case of $\varphi>2$, ν_1^* behaves the same way, but the switching rate ν_2^* corresponding to the second reversal of the current tends to infinity at a finite noise amplitude $a_0=a_c(q)$—at greater amplitudes there is only one reversal of the current. As $q\to 1/2$, the value of ν^* corresponding to the critical amplitude $a_2(q)$ decreases and saturates at a finite value ν_0^*. The exact value of the parameter ν_0^* is given by the solution of Eq. (38) with $\nu_0^*=x$.

In the case of a large potential asymmetry, $b\gg c$, the value of ν_0^* can be estimated by the following equation:

$$\nu_0^*\approx\sqrt{2}bc\left(1+\frac{\sqrt{2}c}{3b}+\frac{11c^2}{36b^2}\right). \tag{50}$$

Though in general, the zero points $\nu_{1,2}^*$ of the current $J(\nu)$ cannot be expressed by elementary functions, at certain constraints rather simple approximate solutions can be found for them. Here we give two such formulas, neither of them applicable near the phase boundary (b), $a_0\gg a_2(q)$.

The first reversal point of the current can be given as

$$\nu_1^*\approx\frac{2b^2c^2}{(1-2q)a_0^2}\left[1+\frac{b^2+c^2}{a_0^2(1-2q)}\left(1-\frac{2q}{3}\right)\right]. \tag{51}$$

The aforementioned monotonic decrease of ν_1^* at a growth of either a_0 or φ can be deduced easily. At fixed a_0 and q it can be seen that ν_1^* increases as the potential asymmetry grows, i.e., as d decreases [see Eq. (25)].

For the estimation of the high values of the second reversal point of the current the following equation would do:

$$\nu_2^*\approx\frac{b^2c^2(a_0^2-b^2)(a_0^2-c^2)\ln[(1+qa_0/\gamma)/(1+qa_0/\eta)]}{c^2(a_0^2-c^2)(b^2+a_0\eta-qa_0^2)-b^2(a_0^2-b^2)(c^2+a_0\gamma-qa_0^2)}, \tag{52}$$

with $\nu^*\gg b(a_0+b)$. Evidently, Eq. (52) is applicable only in region (3) of the phase space (see Fig. 2). If condition $a_0\gg b/q$ is also fulfilled, Eq. (52) can be given a more transparent form:

$$\nu_2^*\approx 2qa_0^2(1-2q)[4q-1+(b^2+c^2)(6q-1)/8q^2a_0^2]^{-1}. \tag{53}$$

The monotonic growth of ν_2^* at the growth of both noise amplitude and flatness parameter immediately follows from Eq. (53), where ν_2^* drops if the potential asymmetry grows.

In the vicinity of the critical lines (b) and (c) (see Fig. 2) we can see that the dependence of ν_1^* and ν_2^* on the noise amplitude has obtained some formally similar features to the second kind phase transitions, e.g., if $q\to 1/2$, $b\gg c$ and $a_0^2\to a_2^2$, $a_0^2>a_2^2$, then

$$\nu_{1,2}^*-\nu_0^*\approx\pm\sqrt{\frac{2\nu_0^*(b^2+\nu_0^*)}{a_2^2}(a_0^2-a_2^2)}\sim(a_0^2-a_2^2)^{1/2},$$

where the sign + is for ν_2^* and the sign − is for ν_1^*. If $1/4>q\to 1/4$ and $a_0^2\to a_c^2$, $a_0^2<a_c^2$, then

$$\nu_2^*\approx\frac{qa_c^6}{(b^2+c^2)(a_c^2-a_0^2)}\sim(a_c^2-a_0^2)^{-1}.$$

Most likely, the different values of the "critical indices" indicate different physical mechanisms for the phase transitions at the critical phase lines (b) and (c).

VI. CONCLUDING REMARKS

Above, we have presented some analytical and exact results for the dynamics of an overdamped Brownian particle in a sawtooth ratchet potential subjected to external colored trichotomous fluctuations. A major virtue of the models with trichotomous noise is that they constitute another case admitting an exact analytical solution for the stationary current for any value of the correlation time $\tau_c=1/\nu$, the noise amplitude a_0, and the flatness parameter φ. Although both dichotomous and trichotomous noises may be too rough approximations in most practical cases, the latter is more flexible, including all cases of dichotomous noises and, as such, revealing the essence of its peculiarities.

The behavior of the system, the current reversals considered, is dominated by the correlation time. In the phase space of the parameters φ, a_0 one can distinguish between four domains of qualitatively different shapes of the current $J(\nu)$, characterized also by sign reversals (Fig. 2). Our major results are perhaps the exact conditions for the noise parameters leading to the sign reversals of J [Eq. (49)]. Three circumstances should be pointed out at that: (i) there is a lower limit for the noise amplitude, namely $a_0=b+c$, for smaller values of which there is no current reversal at any τ_c and φ; (ii) the correlation time has an upper limit $\tau_c=1/\nu_0^*$, where ν_0^* is the solution of Eq. (38), for greater values of which there cannot be more than one current reversal; (iii) the flatness parameter has a critical value $\varphi=2$—if $\varphi<2$, then, as the correlation time grows from 0 to ∞, there can be either two reversals or none, and if $\varphi>2$, there can also occur one reversal. For both slow and fast fluctuating forces we have presented approximations, which agree with the results of [11,17]. It is remarkable that at sufficiently large noise amplitudes, $a_0^2 \gg \max\{b^2/q^2(1-2q),\ \nu/2q(1-2q)\}$, the behavior of the current is completely due to the effect of the ''flashing barrier'' for all values of the correlation time and the flatness parameter. It should be noted that in earlier papers [17,19] the flashing barrier effect of the current has been evaluated only at the adiabatic limit.

We envisage possible applications of the described current reversal phenomena in natural sciences such as biophysics and microtechnology. Particles with different damping constants moving in the same potential and driven by the same stochastic force are controled by different effective τ_c [see Eq. (7)]. This can lead to an efficient mechanism for separating particles as suggested in [1,5,9,17]. Examination of the curves in Fig. 1 shows that two regimes of extreme sensitivity to noise parameters might be applied for separation purposes: for the correlation times $\tau_1=1/\nu_1^*$ and $\tau_2=1/\nu_2^*$, where ν_1^* and ν_2^* are solutions of Eq. (49), the current reversals lead to rather selective behaviors. Note that Eq. (49) enables one to find τ_1 and τ_2 with any precision. Finally, details known about the solutions of Eq. (14) can be of use in testing approximate methods in the theory of stochastic differential equations.

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